

A set of the Viète-like recurrence relations for the unity constant

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Abstract

Using a simple Viète-like formula for π based on the nested radicals $a_k = \sqrt{2 + a_{k-1}}$ and $a_1 = \sqrt{2}$, we derive a set of the recurrence relations for the constant 1. Computational test shows that application of this set of the Viète-like recurrence relations results in a rapid convergence to unity.

Keywords: arctangent function, constant pi, constant 1

1 Description and implementation

1.1 Derivation

Several centuries ago the French mathematician François Viète derived a remarkable formula for pi

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \frac{\sqrt{2 + \sqrt{2}}}{2} \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \dots \quad (1)$$

Nowadays this well-known equation is commonly regarded as the Viète's formula for pi [1, 2, 3, 4]. The uniqueness of this formula is due to nested

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radicals consisting of square roots of twos only. Defining these nested radicals as

$$\begin{aligned}
a_1 &= \sqrt{2}, \\
a_2 &= \sqrt{2 + \sqrt{2}}, \\
a_3 &= \sqrt{2 + \sqrt{2 + \sqrt{2}}} \\
&\vdots \\
a_k &= \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}_{k \text{ square roots}}
\end{aligned}$$

the Viète's formula (1) for pi can be rewritten in a compact form as follows

$$\frac{2}{\pi} = \lim_{k \rightarrow \infty} \prod_{k=1}^K \frac{a_k}{2}.$$

There is a simple Viète-like formula for pi that can be represented in form [5]

$$\frac{\pi}{2^{k+1}} = \arctan\left(\frac{\sqrt{2 - a_{k-1}}}{a_k}\right), \quad k \geq 2, \quad (2)$$

From this formula it follows that

$$\begin{aligned}
\frac{\pi}{2^3} + \frac{\pi}{2^4} + \frac{\pi}{2^5} \cdots = \\
\arctan\left(\frac{\sqrt{2 - a_1}}{a_2}\right) + \arctan\left(\frac{\sqrt{2 - a_2}}{a_3}\right) + \arctan\left(\frac{\sqrt{2 - a_3}}{a_4}\right) + \cdots
\end{aligned} \quad (3)$$

and because of the decreasing geometric series

$$\frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} \cdots = \frac{1}{4}$$

the equation (3) can be expressed in a more simplified form

$$\frac{\pi}{4} = \lim_{K \rightarrow \infty} \sum_{k=1}^K \arctan\left(\frac{\sqrt{2 - a_k}}{a_{k+1}}\right). \quad (4)$$

It is more convenient for our purpose to represent the equation (4) as

$$\begin{aligned} \frac{\pi}{4} = & \arctan \left(\frac{\sqrt{2 - \sqrt{2}}}{\sqrt{2 + \sqrt{2}}} \right) + \arctan \left(\frac{\sqrt{2 - \sqrt{2 + \sqrt{2}}}}{\sqrt{2 + \sqrt{2 + \sqrt{2}}}} \right) \\ & + \arctan \left(\frac{\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}{\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} \right) + \dots \end{aligned}$$

or

$$\begin{aligned} \frac{\pi}{4} &= \arctan(b_1) + \arctan(b_2) + \arctan(b_3) \dots \\ &= \lim_{K \rightarrow \infty} \sum_{k=1}^K \arctan(b_k), \end{aligned}$$

where the arguments of the arctangent functions can be found by using the recurrence relations

$$b_k = \frac{\sqrt{2 - a_k}}{a_{k+1}}$$

and

$$a_k = \sqrt{2 + a_{k-1}}, \quad a_1 = \sqrt{2}.$$

Since

$$\arctan(1) = \frac{\pi}{4}$$

we can also write

$$\arctan(1) = \lim_{K \rightarrow \infty} \sum_{k=1}^K \arctan(b_k). \quad (5)$$

The right side of the equation (5) consists of the infinite summation terms of the arctangent functions. We may attempt to exclude the infinite sum using the identity

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{x + y}{1 - xy}\right) \quad (6)$$

repeatedly. Specifically, we employ the following recurrence relations that just reflects the successive application of the identity (6) above

$$c_k = \frac{c_{k-1} + b_k}{1 - c_{k-1}b_k}, \quad c_1 = b_1.$$

This enables us to rewrite the equation (5) as

$$\arctan(1) = \arctan(c_k) + \lim_{L \rightarrow \infty} \sum_{\ell=k+1}^L \arctan(b_\ell). \quad (7)$$

According to the Maclaurin expansion series

$$\arctan(b_\ell) = b_\ell - \frac{b_\ell^3}{3} + \frac{b_\ell^5}{5} - \frac{b_\ell^7}{7} + \dots = b_\ell + O(b_\ell^3).$$

Since at $\ell \rightarrow \infty$ the variable $b_\ell \rightarrow 0$ and, therefore, due to negligible $O(b_\ell^3)$ we can simply replace it by $\arctan(b_\ell)$ and then use the equation (2) in order to find a ratio of the limit

$$\lim_{\ell \rightarrow \infty} \frac{b_{\ell+1}}{b_\ell} = \lim_{\ell \rightarrow \infty} \frac{\arctan(b_{\ell+1})}{\arctan(b_\ell)} = \lim_{\ell \rightarrow \infty} \frac{\pi/2^{\ell+2}}{\pi/2^{\ell+1}} = \frac{1}{2}. \quad (8)$$

Consider the following infinite sequence

$$\{b_1, b_2, b_3, \dots, b_\ell, \dots\}. \quad (9)$$

According to the limit (8) the ratio $b_{\ell+1}/b_\ell$ tends to $1/2$ with increasing index ℓ . Consequently, it is not difficult to see now that

$$\frac{b_2}{b_1} < \frac{b_3}{b_2} < \frac{b_4}{b_3} < \dots < \frac{b_{\ell+1}}{b_\ell} < \dots < \frac{1}{2}.$$

In fact, the tendency of the ratio $b_{\ell+1}/b_\ell$ towards $1/2$ with increasing index ℓ is very fast. In particular, when the index ℓ is large enough, say at $\ell > 10$, the sequence (9) behaves almost like a decreasing geometric progression where a common ratio is $1/2$.

Since the index k in the equation (7) can be taken arbitrarily large, we can rewrite it in form

$$\arctan(1) = \lim_{k \rightarrow \infty} \left[\arctan(c_k) + \lim_{L \rightarrow \infty} \sum_{\ell=k+1}^L b_\ell \right]. \quad (10)$$

Taking into account that the ratio $b_{\ell+1}/b_\ell$ tends to but never exceeds $1/2$, we can conclude that the damping rate in the sequence (9) is faster than that of in a decreasing geometric progression

$$\left\{ b_1, \frac{b_1}{2}, \frac{b_1}{2^2}, \frac{b_1}{2^3}, \dots, \frac{b_1}{2^\ell} \dots \right\}$$

with fixed common ratio $1/2$. This signifies that

$$\sum_{\ell=k+1}^L b_{\ell} < \sum_{\ell=k+1}^L \frac{b_1}{2^{\ell-1}}, \quad L > k > 0,$$

and since the limit of the decreasing geometric series

$$\lim_{L \rightarrow \infty} \sum_{\ell=k+1}^L \frac{b_1}{2^{\ell-1}} \rightarrow 0, \quad k \rightarrow \infty,$$

we prove that

$$\lim_{L \rightarrow \infty} \sum_{\ell=k+1}^L b_{\ell} \rightarrow 0, \quad k \rightarrow \infty.$$

As a consequence, the equation (10) can be further simplified as

$$\arctan(1) = \lim_{k \rightarrow \infty} \arctan(c_k) \Leftrightarrow 1 = \lim_{k \rightarrow \infty} c_k.$$

Thus, we can infer that the constant 1 can be approached successively by increment of the index k in a set of the Viète-like recurrence relations

$$\left\{ \begin{array}{l} a_1 = \sqrt{2}, \\ a_k = \sqrt{2 + a_{k-1}}, \\ b_k = \frac{\sqrt{2 - a_k}}{a_{k+1}}, \\ c_1 = b_1, \\ c_k = \frac{c_{k-1} + b_k}{1 - c_{k-1}b_k}, \end{array} \right. \quad (11)$$

such that $c_{k \rightarrow \infty} \rightarrow 1$.

1.2 Computation

Consider the first three elements from the sequence (9)

$$b_1 = \frac{\sqrt{2 - a_1}}{a_2} = \frac{\sqrt{2 - \sqrt{2}}}{\sqrt{2 + \sqrt{2}}},$$

$$b_2 = \frac{\sqrt{2-a_2}}{a_3} = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

and

$$b_3 = \frac{\sqrt{2-a_3}}{a_4} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}.$$

Consequently, the corresponding first three values of the variable c_k are

$$c_1 = b_1 = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} = 0.41421356237309504880\dots,$$

$$c_2 = \frac{c_1 + b_2}{1 - c_1 b_2} = \frac{\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} + \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}}{1 - \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}} = 0.66817863791929891999\dots$$

and

$$\begin{aligned} c_3 = \frac{c_2 + b_3}{1 - c_2 b_3} &= \frac{\frac{\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} + \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}}{1 - \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}} + \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{1 - \frac{\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} + \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}}{1 - \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}} \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}} \\ &= 0.82067879082866033097\dots, \end{aligned}$$

respectively.

From these examples one can see that the set (11) of the Viète-like recurrence relations gradually builds the continued fractions in the numerator and denominator of the variable c_k at each successive step in increment of the index k . It is also interesting to note that each value of the variable c_k is based on nested radicals consisting of square roots of twos only.

Figure 1 shows the dependence of the variables a_k , b_k and c_k as a function of the index k by blue, green and red colors, respectively. We can observe

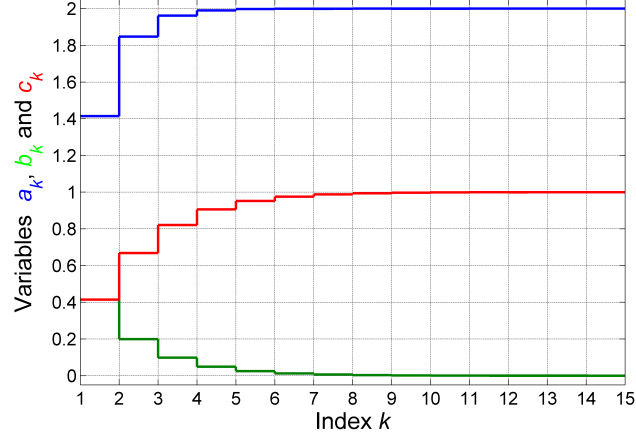


Fig. 1. Evolution of the variables a_k (blue), b_k (green) and c_k (red).

how the variable c_k tends to 1 while the variables a_k and b_k tend to 2 and 0, respectively.

Table 1 shows the values of variable c_k and error term $\epsilon_k = 1 - c_k$ with corresponding index k ranging from 4 to 15. As we can see from this table, the variable c_k quite rapidly tends to unity with increasing index k . In particular, the error term ϵ_k decreases by factor of about 2 at each increment of the index k by one.

Table 1. The variable c_k and error term ϵ_k at index k ranging from 4 to 15.

k	c_k	ϵ_k
4	0.90634716901914715794...	0.09365283098085284205...
5	0.95207914670092534858...	0.04792085329907465141...
6	0.97575264993237653232...	0.02424735006762346767...
7	0.98780284145152917070...	0.01219715854847082929...
8	0.99388282491415211156...	0.00611717508584788843...
9	0.99693673501114949604...	0.00306326498885050395...
10	0.99846719455859369106...	0.00153280544140630893...
11	0.99923330359286120490...	0.00076669640713879509...
12	0.99961657831851611515...	0.00038342168148388484...
13	0.99980827078273533526...	0.00019172921726466473...
14	0.99990413079635610519...	0.00009586920364389480...
15	0.99995206424931502866...	0.00004793575068497133...

2 New formula for pi

As the error term ε_k decreases successively by factor of about 2 (see third column in the Table 1), we may expect that $2^k \varepsilon_k$ is convergent and tends to some constant when the index k tends to infinity. The computational test shows that the value $2^k \varepsilon_k$ approaches to $\pi/2$ as the index k increases. Therefore, we assume that

$$\lim_{k \rightarrow \infty} 2^k \varepsilon_k = \frac{\pi}{2}$$

or

$$\pi = \lim_{k \rightarrow \infty} 2^{k+1} (1 - c_k).$$

Furthermore, relying on numerical results we also suggest a generalization to the power m as given by

$$m \pi = \lim_{k \rightarrow \infty} 2^{k+1} (1 - c_k^m). \quad (12)$$

Since the variable c_k is determined within the set (11) of the Viète-like recurrence relations, the new equation (12) can also be regarded as the Viète-like formula for pi.

3 Conclusion

We show a set (11) of the Viète-like recurrence relations for the constant 1 derived by using the Viète-like formula (2) for pi. Sample computations reveal that the variable c_k quite rapidly tends to unity as the index k increases.

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